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Grouping of products of wavefunctions and Gell-Mann dynamic symmetry

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Abstract. The dynamical equations of evolution of the products of components of the wavevector of an N-level system possessing the so-called Gell-Mann dynamic symmetry are studied for two sets of products which resemble the meson and baryon wavefunctions in elementary particle physics. The equations are shown to decompose into various independent subsets which resemble the groupings of the corresponding elementary particles. It is emphasised that dynamical consideration is the principle used in our grouping scheme. The sets of product wavefunctions, regardless of their physical origin, will fall into groups given in this paper if the dynamics of the wavevector of the N-level system has the symmetry of the Gell-Mann type defined.

1. Introduction

The problem of dynamic symmetry in quantum electronics was studied by the author in a series of papers in recent years [1-6]. A system consisting of atoms or molecules with N transition levels interacting with an intense laser field is said to possess a dynamic symmetry if the system possesses constants of evolution other than the constant of total population (in a time short compared with the natural decay times of the excitations). In particular, the principal features of, and the conditions for, a system to possess the so-called Gell-Mann dynamic symmetry were found [4]. It was shown that the characteristic set of constants of evolution which the system possesses in this case closely resembles the set of quantum numbers associated with the isospin invariance, strangeness, charm, bottom and top etc, in elementary particle physics (see, for example, [7]).

Let the N components of the column wavevector $\boldsymbol{\Psi}(t) = \mathrm{col}(\Psi_1(t), \Psi_2(t), \ldots, \Psi_N(t))$ represent the probability amplitudes of the system in the N corresponding levels or states at time t. If the system possesses the Gell-Mann symmetry, then the characteristic set of constants of evolution can be expressed in the following form [4]:

$$|\boldsymbol{u}_1^\dagger \cdot \boldsymbol{\Psi}(t)|^2 + |\boldsymbol{u}_2^\dagger \cdot \boldsymbol{\Psi}(t)|^2 = \text{constant} \tag{1.1a}$$

$$|\boldsymbol{u}_m^\dagger \cdot \boldsymbol{\Psi}(t)| = \text{constant} \qquad m = 3, 4, \ldots, N \tag{1.1b}$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N$ are a set of N-dimensional column vectors which depend on the interaction parameters in the Hamiltonian of the system, and $\boldsymbol{u}_1^\dagger, \boldsymbol{u}_2^\dagger, \ldots, \boldsymbol{u}_N^\dagger$ denote

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their corresponding complex conjugate row vectors. Alternatively, we can express (1) in terms of the transformed wavevector $\boldsymbol{\psi}(t)=(\psi_1(t), \psi_2(t), \ldots, \psi_N(t))$ as

$$|\psi_1(t)|^2+|\psi_2(t)|^2=\text{constant} \tag{1.2a}$$

$$|\psi_m(t)|=\text{constant} \qquad m=3,4,\ldots,N \tag{1.2b}$$

where $\boldsymbol{\psi}(t)$ is related to $\boldsymbol{\Psi}(t)$ by

$$\boldsymbol{\psi}(t)=\hat{U}^{+}\boldsymbol{\Psi}(t) \qquad \text{or} \qquad \boldsymbol{\Psi}(t)=\hat{U}\boldsymbol{\psi}(t) \tag{1.3}$$

$\hat{U}$ being the matrix whose columns are made up of the column vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N$.

Thus, an important consequence of the Gell-Mann symmetry is that the space in which the transformed wavevector $\boldsymbol{\psi}(t)$ evolves decomposes into independent subspaces. The time-dependent Schrödinger equation which $\boldsymbol{\psi}(t)$ satisfies is

$$\mathrm{i}\hbar\frac{\partial\boldsymbol{\psi}}{\partial t}=\hat{\mathscr{H}}(t)\boldsymbol{\psi} \tag{1.4a}$$

where $\hat{\mathscr{H}}(t)$ is a block diagonal matrix of the form

$$\hat{\mathscr{H}}(t)=-\hbar\left[\begin{array}{cc|cc|cc|c} 0 & h_{12}(t) & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & \\ h_{12}^*(t) & \Delta(t) & & & & & \\ \hline \multicolumn{2}{c|}{0} & 0 & & \multicolumn{2}{c|}{0} & \\ & & & \Delta(t) & & & \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 & & \\ & & & & & \Delta(t) & \\ \hline & & & & & & \ddots \end{array}\right]. \tag{1.4b}$$

A system whose time evolution obeys the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\boldsymbol{\Psi}}{\partial t}=\hat{H}(t)\boldsymbol{\Psi} \tag{1.5}$$

is said to possess the Gell-Mann dynamic symmetry if a time-independent unitary matrix $\hat{U}$ can be found such that the Hamiltonian $\hat{H}$ of the system can be transformed through (1.3) to (1.4). A particularly interesting class of physical Hamiltonian $\hat{H}(t)$ which possess the Gell-Mann symmetry was given in [4], and has matrix elements $H_{jk}(t)$ as follows:

$$\text{for } j\neq k \qquad H_{jk}(t)=\begin{cases} 0 & \text{for } |j-k|=\text{even number} \\ a_j a_k^* f(t) & \text{for } |j-k|=\text{odd number}, j \text{ odd} \\ a_j a_k^* f^*(t) & \text{for } |j-k|=\text{odd number}, j \text{ even} \end{cases}$$

$$H_{jj}(t)=\begin{cases} 0 & \text{for } j \text{ odd} \\ g(t) & \text{for } j \text{ even} \end{cases}$$

where $j, k=1,2,\ldots,N$, $f(t)$ and $g(t)$ are any arbitrary time-dependent functions, and a_j any arbitrary constants.

Equations (1.4) showed that the components of the transformed wavevector $\boldsymbol{\psi}(t)$ can be characterised and grouped according to the dynamics of the system. Suppose we now form products $\psi_j\psi_k\psi_l\ldots$ of these wavevector components where the components of different subscripts need not commute, or more generally, we form products in which every wavevector component also has a spin component s attached to it which may be spin up ($\uparrow$) or spin down ($\downarrow$). We assume that $\psi_n^{(s)}(t)$ obeys (1.4) in which the matrix elements of $\hat{\mathscr{H}}(t)$ are independent of the spin s. We now ask how the set of products $\psi_j^{(s)}\psi_k^{(s')}\psi_l^{(s'')}\ldots$ will be divided and grouped into subsets which evolve independently of each other. In analogy with elementary particle physics, the sets of products whose dynamics we shall look into are the set of 'mesons' which consist of various symmetrised and antisymmetrised combinations of $\psi_j^{(s)}\bar{\psi}_j^{(s')}$, where $\bar{\psi}$ denotes the complex conjugate of ψ, and the set of 'baryons' which consist of various combinations of $\psi_j^{(s)}\psi_k^{(s')}\psi_l^{(s'')}$. In quantum electronics, it was the study of the time evolution of the (N^2-1)-dimensional coherence vector [8], whose components are made up of various combinations of $\psi_j\bar{\psi}_k$, which first led us to a grouping scheme which was noticed to resemble that for the pseudoscalar mesons in particle physics [1].

In this paper, we shall give the results on how the $(2N)^2$ 'mesons', which consist of N^2-1 'pseudoscalar mesons', $3N^2$ 'vector mesons' and 1 'singlet', are grouped into subsets according to their dynamics, and we shall also present the grouping scheme for the $(2N)^3$ 'baryons'. We should point out that the decompositions from the group theoretical consideration given by

$$N\otimes N^* = 1\oplus(N^2-1) \tag{1.6}$$

and

$$N\otimes N\otimes N = \frac{(N+2)(N+1)N}{6}\oplus\frac{(N+1)N(N-1)}{3}\oplus\frac{(N+1)N(N-1)}{3}\oplus\frac{N(N-1)(N-2)}{6} \tag{1.7}$$

give only the first step, which merely separates out the product functions according to their symmetry property. Our grouping schemes resulting from the *dynamical considerations* give further subgrouping according to how the set of equations of evolution for these product wavefunctions decomposes into independent subsets, and as we shall see in the following sections, the results of our *dynamical* grouping agree with the grouping of the elementry particles according to the theory of quarks. It is not the purpose of this paper to speculate on whether (1.4) have any direct relevance to the dynamics of the actual quarks, but we are simply presenting the results as they are. The quantities represented by the combinations of the products of ψ_j may arise in other physical problems and have other physical interpretations (such as in the case of quantum electronics already mentioned [1, 8]). If $\boldsymbol{\psi}$ satisfies an equation of the form (1.4), then the equations of evolution for the sets of products of ψ_j decompose into independent subsets as our results summarised in tables 1 and 3 show.

The components $\psi_1, \psi_2, \ldots, \psi_N$ of $\boldsymbol{\psi}$ given by (1.4) are the analogues of the quark wavefunctions, and their complex conjugates $\bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_n$ are the analogues of the antiquark wavefunctions, where N corresponds to the number of quark flavours. The odd and even subscripts of the wavefunctions divide naturally into two classes from (1.4); the quark wavefunctions u, d, s, c, b, t, ... correspond to the wavefunctions

Table 1. The grouping of the wavefunctions for the 'pseudoscalar mesons' $\phi(\mathrm{S})$ (the upper sign) and for the 'vector mesons' $\phi(\mathrm{A})$ (the lower sign); $m, n = 3, 4, \ldots, N$.

Nature of group	Elements
1 group of 3	$(2)^{-1/2}(\psi_1\bar{\psi}_2 \pm \bar{\psi}_2\psi_1)$, $\frac{1}{2}[(\psi_1\bar{\psi}_1 - \psi_2\bar{\psi}_2) \pm (\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2)]$, $(2)^{-1/2}(\bar{\psi}_1\psi_2 \pm \psi_2\bar{\psi}_1)$
$N-2$ groups of 4	$(2)^{-1/2}(\psi_1\bar{\psi}_m \pm \bar{\psi}_m\psi_1)$, $(2)^{-1/2}(\psi_2\bar{\psi}_m \pm \bar{\psi}_m\psi_2)$, $(2)^{-1/2}(\bar{\psi}_1\psi_m \pm \psi_m\bar{\psi}_1)$, $(2)^{-1/2}(\bar{\psi}_2\psi_m \pm \psi_m\bar{\psi}_2)$.
$\frac{1}{2}(N-2)(N-3)$ groups of 2	$(2)^{-1/2}(\psi_m\bar{\psi}_n \pm \bar{\psi}_n\psi_m)$, $(2)^{-1/2}(\bar{\psi}_m\psi_n \pm \psi_n\bar{\psi}_m)$, $m \neq n$
$N-2$ groups of 1	$(12)^{-1/2}[(\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 - 2\psi_3\bar{\psi}_3) \pm (\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 - 2\bar{\psi}_3\psi_3)]$, $(24)^{-1/2}[(\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 + \psi_3\bar{\psi}_3 - 3\psi_4\bar{\psi}_4) \pm (\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 + \bar{\psi}_3\psi_3 - 3\bar{\psi}_4\psi_4)]$, ... $(2N(N-1))^{-1/2}\left[\left(\sum_{j=1}^{N-1} \psi_j\bar{\psi}_j - (N-1)\psi_N\bar{\psi}_N\right) \pm \left(\sum_{j=1}^{N-1} \bar{\psi}_j\psi_j - (N-1)\bar{\psi}_N\psi_N\right)\right]$
Extra group of 1 for the 'vector meson'	$(2N)^{-1/2} \sum_{j=1}^{N} (\psi_j\bar{\psi}_j - \bar{\psi}_j\psi_j)$
Singlet	$(2N)^{-1/2} \sum_{j=1}^{N} (\psi_j\bar{\psi}_j + \bar{\psi}_j\psi_j)$

Table 2. The symmetric (S), mixed symmetric ($\mathrm{M_S}$), mixed antisymmetric ($\mathrm{M_A}$), and antisymmetric (A) arrangments of 'quark flavour' wavefunctions α, β , γ.

Arrangement	Wavefunctions
S	$\phi_1(\mathrm{S}) = \alpha\alpha\alpha$
	$\phi_2(\mathrm{S}) = (3)^{-1/2}(\alpha\beta\alpha + \beta\alpha\alpha + \alpha\alpha\beta)$
	$\phi_3(\mathrm{S}) = 6)^{-1/2}[(\beta\gamma + \gamma\beta)\alpha + (\gamma\alpha + \alpha\gamma)\beta + (\alpha\beta + \beta\alpha)\gamma]$
$\mathrm{M_S}$	$\phi_1(\mathrm{M_S}) = (6)^{-1/2}[(\alpha\beta + \beta\alpha)\alpha - 2\alpha\alpha\beta]$
	$\phi_2(\mathrm{M_S}) = (12)^{-1/2}[(\beta\gamma + \gamma\beta)\alpha + (\gamma\alpha + \alpha\gamma)\beta - 2(\alpha\beta + \beta\alpha)\gamma]$
	$\phi_3(\mathrm{M_S}) = \frac{1}{2}[(\beta\gamma + \gamma\beta)\alpha - (\gamma\alpha + \alpha\gamma)\beta]$
$\mathrm{M_A}$	$\phi_1(\mathrm{M_A}) = (2)^{-1/2}(\alpha\beta - \beta\alpha)\alpha$
	$\phi_2(\mathrm{M_A}) = \frac{1}{2}[-(\beta\gamma - \gamma\beta)\alpha + (\gamma\alpha - \alpha\gamma)\beta]$
	$\phi_3(\mathrm{M_A}) = (12)^{-1/2}[-(\beta\gamma - \gamma\beta)\alpha - (\gamma\alpha - \alpha\gamma)\beta + 2(\alpha\beta - \beta\alpha)\gamma]$
A	$\phi(\mathrm{A}) = (6)^{-1/2}[(\beta\gamma - \gamma\beta)\alpha + (\gamma\alpha - \alpha\gamma)\beta + (\alpha\beta - \beta\alpha)\gamma]$

given in the parentheses in the following:

$$
\begin{array}{cccc}
\mathrm{u}(\psi_2) & \mathrm{c}(\psi_4) & \mathrm{t}(\psi_6) & \ldots \\
\mathrm{d}(\psi_1) & \mathrm{s}(\psi_3) & \mathrm{b}(\psi_5) & \ldots .
\end{array} \tag{1.8}
$$

The d and u quark wavefunctions, which correspond to wavefunctions with subscripts 1 an 2, obey a set of coupled equations somewhat different from those for the remaining quark wavefunctions with subscripts greater than 2. Unless otherwise specified, the subscripts m, n, p will be used in the following to denote numbers greater than 2 (and $\leqslant N$). We shall denote by S the symmetric arrangement of quark flavour or spin

Table 3. The grouping of the 'baryons'; $m, n, p = 3, 4, \ldots, N$.

S (total number $= \sigma_{N+2}$)	
1 group of 4	$(\psi_1\psi_1\psi_1)$, $(\psi_1\psi_1\psi_2)$, $(\psi_1\psi_2\psi_2)$, $(\psi_2\psi_2\psi_2)$
$N-2$ groups of 3	$(\psi_1\psi_1\psi_m)$, $(\psi_1\psi_2\psi_m)$, $(\psi_2\psi_2\psi_m)$
$\frac{1}{2}(N-2)(N-1)$ groups of 2	$(\psi_m\psi_n\psi_1)$, $(\psi_m\psi_n\psi_2)$
$\frac{1}{6}(N-2)(N-1)N$ groups of 1	$(\psi_m\psi_n\psi_p)$
M_S or M_A (total number $= 2\sigma_{N+1}$)	
1 group of 2	$(\psi_1\psi_1\psi_2)$, $(\psi_2\psi_2\psi_1)$
$N-2$ groups of 4	$(\psi_1\psi_1\psi_m)$, $(\psi_1\psi_2\psi_m)$, $(\psi_1\psi_2\psi_m)'$, $(\psi_2\psi_2\psi_m)$
$(N-2)^2$ groups of 2	$(\psi_m\psi_n\psi_1)$, $(\psi_m\psi_n\psi_2)$
$\frac{1}{3}(N-3)(N-2)(N-1)$ groups of 1	$(\psi_m\psi_n\psi_p)$
A (total number $= \sigma_N$)	
$N-2$ groups of 1	$(\psi_1\psi_2\psi_m)$
$\frac{1}{2}(N-2)(N-3)$ groups of 2	$(\psi_m\psi_n\psi_1)$, $(\psi_m\psi_n\psi_2)$, $m \neq n$
$\frac{1}{6}(N-2)(N-3)(N-4)$ groups of 1	$(\psi_m\psi_n\psi_p)$, $m \neq n \neq p$

wavefunctions, by A the antisymmetric arrangement and by M_S and M_A the mixed symmetric and mixed antisymmetric arrangements.

2. Mesons

The N^2-1 symmetric arrangements of $\psi_j\bar{\psi}_k$ given in table 1, which will be collectively called $\phi(S)$, are the analogues of, and will be referred to as, the quark flavour part of the wavefunctions of the pseudoscalar mesons. The N^2 antisymmetric arrangements of $\psi_j\bar{\psi}_k$ given in table 1, which will be collectively called $\phi(\mathrm{A})$, are the analogues of, and will be referred to as, the quark flavour part of the wavefunctions of the vector mesons. The wavefunctions of the singlet is also given in table 1. The grouping of these wavefunctions or 'particles' into groups of 3, 4, 2, and 1 as presented in table 1 can be verified by differentiating the wavefunctions with respect to the time, assuming that ψ_j and $\bar{\psi}_k$ do not generally commute, and by using (1.4). For example, for $N=4$ (four quark flavours d, u, s, c), the equations of evolution for the pseudoscalar mesons decompose into the following independent subsets of groups: 1 group of 3 (π^-, π^0, π^+), 2 groups of 4 (K^0, K^+, $\bar{K}^0$, K^- and D^0, D^+, $\bar{D}^0$, D^-), 1 group of 2 (F^-, F^+) and 2 groups of 1 (η and η_c); the equations of evolution for the vector mesons decompose into 1 group of 3(ρ^-, ρ^0, ρ^+), 2 groups of 4 (K^{0*}, K^{+*}, $\bar{K}^{0*}$, K^{-*} and D^{0*}, D^{+*}, $\bar{D}^{0*}$, D^{-*}), 1 group of 2 (F^{-*}, F^{+*}) and 3 groups of 1 (ω, ϕ and ψ); the singlet is (η_1). The particles given in the parentheses above are the corresponding particles of the actual mesons.

The spin content part of the wavefunctions consists of the following symmetric and antisymmetric arrangements:

$$\mathrm{S}: \quad \chi_1(\mathrm{S}) = \uparrow\uparrow \qquad \chi_2(\mathrm{S}) = (2)^{-1/2}(\uparrow\downarrow + \downarrow\uparrow) \qquad \chi_3(\mathrm{S}) = \downarrow\downarrow \tag{2.1a}$$

$$\mathrm{A}: \quad \chi(\mathrm{A}) = (2)^{-1/2}(\uparrow\downarrow - \downarrow\uparrow). \tag{2.1b}$$

Taking into account the spin content as well as the quark flavour content, the wavefunctions of the mesons are:

$$\text{pseudoscalar mesons } 0^-: \phi(\mathrm{S})\chi(\mathrm{A}) \tag{2.2a}$$

$$\text{vector mesons } 1^-: \phi(\mathrm{A})\chi_i(\mathrm{S}) \qquad i=1,2,3. \tag{2.2b}$$

For example, using table 1 and (1.8), the wavefunctions for the pseudoscalar meson π^+, and the vector meson ρ^+ corresponding to $J_z=1$, are

$$|\pi^+\rangle = \tfrac{1}{2}(\mathrm{u}\uparrow\bar{\mathrm{d}}\downarrow - \mathrm{u}\downarrow\bar{\mathrm{d}}\uparrow + \bar{\mathrm{d}}\uparrow\mathrm{u}\downarrow - \bar{\mathrm{d}}\downarrow\mathrm{u}\uparrow) \tag{2.3a}$$

$$|\rho^+\rangle = (2)^{-1/2}(\mathrm{u}\uparrow\bar{\mathrm{d}}\uparrow - \bar{\mathrm{d}}\uparrow\mathrm{u}\uparrow). \tag{2.3b}$$

Since we have assumed that the matrix elements of $\hat{\mathscr{H}}$ in (1.4b) are independent of the spin, inclusion of the spin would not affect the grouping of particles presented in table 1. Using (2.2) and table 1, we have N^2-1 pseudoscalar mesons, $3N^2$ vector mesons and 1 singlet, making up a total of $(2N)^2$ mesons.

3. Baryons

In this section, we consider the set of N^3 products $\psi_j\psi_k\psi_l$ and see how the set of equations of evolution of these wavefunctions decomposes into independent subsets. In analogy with the baryon wavefunctions in elementary particle physics, we first form combinations of the N^3 product wavefunctions (of three 'quarks') according to their symmetry types. There are the symmetric (S), mixed symmetric ($\mathrm{M_S}$), mixed antisymmetric ($\mathrm{M_A}$) and antisymmetric (A) arrangements of three 'objects'. Depending on whether all three subscripts of ψ are the same (three 'quarks' of the same flavour for which each wavefunction ψ will be denoted by α), or two of three subscripts of ψ are the same (for which the wavefunctions will be denoted by α, α and β), or the ψ have all different subscripts (for which the ψ will be denoted by α, β, γ), the possible arrangements for all the symmetry types are given in table 2. The total number of possible arrangements, ten, is equal to the number of permutations of three objects, which, for the cases of three identical objects ($\alpha\alpha\alpha$), two identical objects ($\alpha\alpha\beta$) and three different objects ($\alpha\beta\gamma$) are given respectively by

$$\frac{3!}{3!}=1 \qquad \frac{3!}{2!1!}=3 \qquad \frac{3!}{1!1!1!}=6 \tag{3.1}$$

where we have made use of the formula for the number of permutations of n things, p of which are of one kind, q of which another etc, given by

$$\frac{n!}{p!q!\ldots}. \tag{3.2}$$

It can be verified that the wavefunctions given in table 2 are orthornormalised.

We shall use S, $\mathrm{M_S}$, $\mathrm{M_A}$ and A to specify the type of wavefunction and write, for example, ($\alpha\alpha\beta$) under the symmetry type S to denote $\phi_2(\mathrm{S})$, ($\alpha\alpha\beta$) under the symmetry type $\mathrm{M_S}$ to denote $\phi_1(\mathrm{M_S})$. However, there are two cases, $\phi_2(\mathrm{M_S})$ and $\phi_3(\mathrm{M_S})$ for ($\alpha\beta\gamma$) under the heading $\mathrm{M_S}$, for which we shall write ($\alpha\beta\gamma$) and ($\alpha\beta\gamma$)$'$. Similarly, we shall write ($\alpha\beta\gamma$) and ($\alpha\beta\gamma$)$'$ under the heading $\mathrm{M_A}$ to denote $\phi_2(\mathrm{M_A})$ and $\phi_3(\mathrm{M_A})$. We shall also use σ_N to denote the combinatorial number

$$\sigma_N = \binom{N}{3} = \tfrac{1}{6}N(N-1)(N-2). \tag{3.3}$$

The results for the grouping of these N^3 wavefunctions, which are the analogues of the baryon wavefunctions in elementary particle physics, are presented in table 3. The subscripts m, n, p are used to denote integers greater than 2 and less than or equal to N. According to the definitions of the notation given above, the wavefunction $(\psi_1\psi_1\psi_m)$ under the heading M_S, for example, denotes $(1/\sqrt{6})[(\psi_1\psi_m+\psi_m\psi_1)\psi_1-2\psi_1\psi_1\psi_m]$ while the wavefunction $(\psi_1\psi_1\psi_m)$ under the heading S denotes $(1/\sqrt{3})(\psi_1\psi_1\psi_m+\psi_1\psi_m\psi_1+\psi_m\psi_1\psi_1)$. The grouping of these wavefunctions, or 'baryons', as presented in table 3 can again be verified by differentiating the wavefunctions with respect to time and by using (1.4).

The total number of wavefunctions for the types S, M_S, M_A and A which are given by σ_{N+2}, $2\sigma_{N+1}$, $2\sigma_{N+1}$ and σ_N respectively are, for the case $N=3$, equal to 10, 8, 8 and 1, which correspond to the familiar baryon decuplet, octets and singlet. The decuplet consists of groups of 4, 3, 2, 1, and the octet consists of groups of 2, 4 and 2 particles. For example, the equations of evolution of S, $(\psi_1\psi_1\psi_1)$, $(\psi_1\psi_1\psi_2)$, $(\psi_1\psi_2\psi_2)$, $(\psi_2\psi_2\psi_2)$, form an independent subset of a group of 4 in the decuplet, while the set involving M_S (or M_A), $(\psi_1\psi_1\psi_2)$, $(\psi_2\psi_2\psi_1)$ (the 'neutron' and the 'proton'), forms a group of 2 in the octet.

For $N=6$, the total numbers of wavefunctions for the types S, M_S, M_A and A are 56, 70, 70 and 20. Our results in table 3 show how the equations of evolution of the wavefunctions or 'particles' in these multiplets decompose into independent subsets or groups for a general value of N.

The spin content part of the wavefunctions consists of the following arrangements:

S: $\chi_1(S)=\uparrow\uparrow\uparrow \qquad \chi_2(S)=(3)^{-1/2}(\uparrow\uparrow\downarrow+\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow)$

$\chi_3(S)=(3)^{-1/2}(\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow+\downarrow\downarrow\uparrow) \qquad \chi_4(S)=\downarrow\downarrow\downarrow$

M_S: $\chi_1(M_S)=(6)^{-1/2}(\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow-2\uparrow\uparrow\downarrow) \qquad \chi_2(M_S)=(6)^{-1/2}(\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow-2\downarrow\downarrow\uparrow)$

M_A: $\chi_1(M_A)=(2)^{-1/2}(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow) \qquad \chi_2(M_A)=(2)^{-1/2}(\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow)$.

Taking account of the spin content as well as the quark flavour content, the wavefunctions of the baryons consist of those products presented in table 4. The total number of baryons, as can be verified, is $(2N)^3$. Again, since we have assumed that

Table 4. The wavefunctions of the baryons, taking account of the spin.

Arrangement	Wavefunctions
S	$\phi_S\chi_S=(\sigma_{N+2},4)$
	$(2)^{-1/2}(\phi_{M_S}\chi_{M_S}+\phi_{M_A}\chi_{M_A})=(2\sigma_{N+1},2)$
M_S	$\phi_S\chi_{M_S}=(\sigma_{N+2},2)$
	$\phi_{M_S}\chi_S=(2\sigma_{N+1},4)$
	$(2)^{-1/2}(-\phi_{M_S}\chi_{M_S}+\phi_{M_A}\chi_{M_A})=(2\sigma_{N+1},2)$
	$\phi_A\chi_{M_A}=(\sigma_N,2)$
M_A	$\phi_S\chi_{M_A}=(\sigma_{N+2},2)$
	$\phi_{MA}\chi_S=(2\sigma_{N+1},4)$
	$(2)^{-1/2}(\phi_{M_S}\chi_{M_A}+\phi_{M_A}\chi_{M_S})=(2\sigma_{N+1},2)$
	$\phi_A\chi_{M_S}=(\sigma_N,2)$
A	$\phi_A\chi_S=(\sigma_N,4)$
	$(2)^{-1/2}(\phi_{M_S}\chi_{M_A}-\phi_{M_A}\chi_{M_S})=(2\sigma_{N+1},2)$

the matrix elements of $\hat{\mathscr{H}}$ in (1.4b) are independent of the spin, inclusion of the spin would not affect the grouping of particles presented in table 3.

4. Summary

A system whose time evolution can be unitarily transformed into a form given by (1.4) is said to possess the Gell-Mann dynamic symmetry. The dynamical equations of evolution for the sets of N^2 products $\psi_j\bar{\psi}_k$ and N^3 products $\psi_j\psi_k\psi_l$ are shown to decompose into various independent subsets. Members of products appearing in the same subsets are grouped. More precisely, various combinations of products as shown in tables 1 and 3 are formed, and they are shown to fall into groups given in the tables according to the dynamical consideration which we stated. If ψ_j, $j = 1, 2, \ldots, N$, are the analogues of N-quark flavour wavefunctions in elementary particle physics, then the various combinations of the product wavefunctions given in tables 1 and 3 are the analogues of the wavefunctions of the mesons and the baryons respectively. The groupings of these wavefunctions agree with the groupings of the corresponding elementary particles. It is tempting to suggest that a Hamiltonian of the form (1.4b) could be the low-energy limit of a more fundamental relativistic field theory in the context of elementary particle physics. In the absence of a clear proof, however, we would leave this possible link to future studies. It is to be stressed that regardless of the origin of the wavefunctions, the sets of product wavefunctions fall into the groups shown in tables 1, 3 and 4 if the dynamics of these functions are characterised by (1.4).

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